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ON THE HAMILTON—OSTROGRADSKII PRINCIPLE IN THE CASE OF IMPULSIVE MOTIONS OF DYNAMIC SYSTEMS*

V.A. SINITSYN

Motion of dynamic systems described by the Hamilton equation and acted upon by the generalized impulsive forces where the impulses have a potential, is studied. The equations of motion are obtained in this case from the condition that the functional of the variational Bolza problem in which the integral part represents a Hamiltonian-type action, is stationary. It is shown that when the impulses are potential, the integral Poincaré—Cartan invariant occurs. Application of the results obtained to the study of the motion of natural systems with discontinuities in the generalized impulses caused by instantaneous changes in the generalized potential, is discussed.

1. Let the motion of a dynamic system be described by the following Hamilton equations:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} + Q_i \quad (i = 1, \ldots, n) \tag{1.1}$$

where H is the Hamilton function of the dynamic system, $q_i, p_i (i = 1, \ldots, n)$ are the canonical variables and Q_i denote the generalized impulsive forces. We denote by S the impulses of the generalized forces over the time of impact τ (the generalized impulses)

$$S_i = \lim_{\substack{Q_i \to \infty \\ \tau \to 0}} \int_0^\tau Q_i \, dt \quad (i = 1, \ldots, n) \tag{1.2}$$

In the presence of the action of instantaneous impulses the equations of impulsive motion become, in accordance with (1.1) and (1.2),

$$q_i^+ - q_i^- = 0, \quad p_i^+ - p_i^- = S_i \quad (i = 1, \ldots, n) \tag{1.3}$$

where the plus and minus signs denote the values of the relevant variable before and after the impulse.

We known that the motion of the system (1.1) over the time intervals during which the impulsive forces Q_i are absent, can be regarded as a continuous sequence of contact transformations generated by the function H. Following this, we shall impose on the generalized impulses the constraints under which the equations of impulsive motion (1.3) become the canonical transformations of the variables q_i and p_i

$$q_i^+ = q_i^-, \quad p_i^+ = p_i^- + S_i \quad (i = 1, \ldots, n) \tag{1.4}$$

The necessary and sufficient conditions for the transformation to be canonical are /1/, that the Lagrange brackets satisfy the following relations:

$$[q_i^- q_k^-] = 0, \quad [p_i^- p_k^-] = 0, \quad [q_i^- p_k^-] = c\delta_{ik} \tag{1.5}$$

where c is the valency of the canonical transformation and δ_{ik} is the Kronecker delta.

Let us write the Lagrange brackets for the transformation (1.4)

$$[q_i^- q_k^-] = \frac{\partial S_i}{\partial q_k^-} - \frac{\partial S_k}{\partial q_i^-}, \quad [p_i^- p_k^-] = 0, \quad [q_i^- p_k^-] = \delta_{ik} + \frac{\partial S_i}{\partial p_k^-} \tag{1.6}$$

From the condition (1.5) it follows for the relations (1.6) that the transformation (1.4) will be canonical if the impulses of the generalized forces satisfy the relations

$$\frac{\partial S_i}{\partial q_k^-} - \frac{\partial S_k}{\partial q_i^-} = 0 \quad (i, k = 1, \ldots, n) \tag{1.7}$$
$$\frac{\partial S_i}{\partial p_k^-} = 0 \quad (i \neq k), \quad 1 + \frac{\partial S_i}{\partial p_k^-} = c \quad (i = k)$$

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Conditions (1.7) imposed on the impulses S_i will of course hold if they are written in the form

$$S_i = (c-1)p_i^- + F_i(q_1^-, \ldots, q_n^-, t) + c_i \tag{1.8}$$

with $\partial F_i / \partial q_k^- = \partial F_k / \partial q_i^-$ $(k, i = 1, \ldots, n)$ Here c_i are constants determined by the conditions (see below) used to define the moment of impulse.

When $c = 1$, (from now on we shall only discuss the impulsive motions corresponding to univalent canonical transformations), the impulses are determined by the function Π which shall be called the impulse potential. Since the impulse field (1.8) is formed as the result of superposition of two fields, we shall also separate from Π the terms linearly dependent on q_i $(i = 1, \ldots, n)$ and t

$$\Pi = -\Pi_0 - \sum_{i=1}^{n} c_i q_i - c_0 t \tag{1.9}$$

Using the potential Π, we find the impulses in the same manner as the forces of the potential force field

$$S_i = -\partial\Pi / \partial q_i^- \quad (i = 1, \ldots, n) \tag{1.10}$$

but the impulse field differs in the fact that its action is localized in time, i.e. an instantaneous application and removal of the field takes place.

Taking (1.9) into account, we write the generating function K of the canonical transformation (1.4) in the form

$$K = \sum_{i=1}^{n} p_i^+ q_i^- + \Pi(q_1^-, \ldots, q_n^-, t) \tag{1.11}$$

Indeed, the generating function (1.11) leads to a transformation corresponding to the impulsive motion (1.4)

$$\frac{\partial K}{\partial q_i^-} = p_i^+ + \frac{\partial \Pi}{\partial q_i^-} = p_i^-, \quad \frac{\partial K}{\partial p_i^+} = q_i^- = q_i^+ \quad (i = 1, \ldots, n)$$
$$H^+ = H^- + \frac{\partial K}{\partial t} = H^- + \frac{\partial \Pi}{\partial t} \tag{1.12}$$

where the independent variables are q_i^-, p_i^+ $(i = 1, \ldots, n)$.

We note that a certain characteristic $X(p, q, t)$ of the dynamic process varies in the impulisve motion in question in such a manner, that the following relation holds for the Poisson brackets /1/ $(XH)^- = (XH)^+$. In particular, the integral independent of time is preserved.

2. We shall show that in the case of the potential impulse field acting on a real motion of a dynamic system, the condition of stationarity holds for the functional constructed in the form of a sum of the function Π and the Hamiltonian action. The following parameters are fixed: the initial and final instant of time t_0 and t_1, the initial and final state of the system, the generalized coordinates (not necessarily all of them), and (or) the instant of the application of the impulses.

An impulsive motion described by the generating function (1.1) begins at the instant of time t^-. We denote by L the Lagrange function and introduce the functional

$$J = -\Pi_0(q_1^-, \ldots, q_n^-, t^-) + \int_{t_0}^{t^-} L\,dt + \int_{t^+}^{t_1} L\,dt \tag{2.1}$$

The instantaneous action of the impulse under which the generalized coordinates remain unchanged as well as fixing certain generalized coordinates and, possibly, the time of application of the impulse, lead to the following relations:

$$\Phi_l = q_l^+ - q_l^- = 0 \quad (l = 1, \ldots, n), \quad \Phi_{n+1} = t^+ - t^- = 0$$
$$\Phi_l = q_{l-n-1}^- - \alpha_{l-n-1} = 0 \quad (l = n+2, \ldots, p < 2n+1),$$
$$\Phi_0 = t^- - \alpha_0 = 0 \tag{2.2}$$

where α_s $(s = 0, 1, \ldots, p-n-1)$ are fixed constants.

The problem of determining the conditions for the minimum of the functional J (2.1) under the constraints (2.2), represents a discontinuous variational problem /2/. The necessary condition of stationarity of the functional and the subsequent passage to canonical variables $(p_i = \partial L) / \partial(dq_i/dt))$ yield the equations (1.1) on the intervals on which the inpulses are absent

$(Q_i = 0)$ and the relations

$$p_i^- = -\frac{\partial\Phi}{\partial q_i^-}, \quad p_i^+ = \frac{\partial\Phi}{\partial q_i^+} \quad (i = 1, \ldots, n) \tag{2.3}$$

$$H^- = \frac{\partial\Phi}{\partial t^-}, \quad H^+ = -\frac{\partial\Phi}{\partial t^+}, \quad \Phi = -\Pi_0 + \sum_{l=0}^{p} \rho_l \Phi_l$$

where $\rho_l\ (l = 0, 1, \ldots, p)$ are undetermined constant multipliers.

Eliminating from (2.3) the undertermined multipliers $\rho_l\ (l = 1, \ldots, n+1)$, we obtain

$$p_i^+ = p_i^- - \frac{\partial \Pi_0}{\partial q_i^-} + \rho_{n+1+i} \quad (i = 1, \ldots, n), \tag{2.4}$$

$$H^+ = H^- + \frac{\partial \Pi_0}{\partial t^-} - \rho_0$$

Comparing (2.4) and (1.12) we see that the constants $c_i\ (i = 0, 1, \ldots, n)$ in (1.9) should be chosen equal to the corresponding undetermined multipliers, i.e.

$$c_i = \rho_{n+1+i}\ (i = 1, \ldots, p-n-1), \quad c_i = 0 \ \text{ when } \ i > p-n-1 \tag{2.5}$$

If the time t^- is not fixed, then $c_0 = 0$ must also be included.

Thus we see that the necessary condition of the stationarity of the functional I holds on the real motion of a dynamic system under the conditions of the Hamilton—Ostrogradskii principle when acted upon by the potential impulses (1.10)

$$\delta I = 0, \quad I = -\Pi + W, \quad W = \int_{t_0}^{t^-} L\, dt + \int_{t^+}^{t_1} L\, dt$$

where W is the Hamiltonian action.

To find the constants $c_i\ (i = 0, 1, \ldots, n)$ appearing in the function Π, we use the second group of the equations of (2.2) and conditions (2.5). The assertion of the Hamilton—Ostrogradskii principle extends naturally to the case of a finite (fixed) number of moments of the action of instantaneous impulses with the potential of the form (1.9).

3. It is clear that in the case of potential impulses the system (1.1) has an integral Poincaré—Cartan invariant. With this in mind we shall consider an extended $(2n+1)$-dimensional phase space of variables q_i, p_i and t. We choose in this space a closed tube of straight paths with the contour C^0 defining the initial state of the system at the time t_0. We draw a curve C^- enveloping the tube and coming in contact with every generatrix once. The contour C^- characterizes the state of the system before the impact and is, generally speaking, arbitrary, since the conditions determining the instant of application of the impulses can be specified in various ways. Assuming that the transformation (1.4) is single-valued, we shall supplement it with the equation $t^+ = t^-$ (instantaneity of the impulse) and construct the contour C^+. When the system is set in motion, the contour defines a new tube of straight paths, and we produce on it an arbitrary closed contour C^1 enveloping the tube.

The two tubes obtained intersect in the subspace $q_i\ (i = 1, \ldots, n)$, t. For each tube we have the integral Poincaré—Cartan invariants /1/.

Let us denote by W_1 and W_2 the Hamilton actions along the generatrices of the tubes from C^0 to C^- and from C^+ to C^1

$$W_1 = \int_{t_0(\alpha)}^{t^-(\alpha)} L\, dt, \quad W_2 = \int_{t^+(\alpha)}^{t_1(\alpha)} L\, dt, \quad L = \sum_i p_i \frac{dq_i}{dt} - H$$

Here L is the Lagrange function written in terms of the canonical variables, and α is a parameter used to represent the equations of the curves in the form

$$q_i = q_i(\alpha), \quad p_i = p_i(\alpha) \quad (i = 1, \ldots, n), \quad t = t(\alpha)$$

We have, for any α /1/

$$\delta W_1 = \Big[\sum_i p_i \delta q_i - H\delta t\Big]_0^-, \quad \delta W_2 = \Big[\sum_i p_i \delta q_i - H\delta t\Big]_+^1$$

Let us find the sum of δW_1 and δW_2, with (2.2) taken into account

$$\delta W = \delta W_1 + \delta W_2 = \Big[\sum_i p_j \delta q_i - H\delta t\Big]_0^1 - \sum_i (p_i^+ - p_i^-)\,\delta q_i^- + (H^+ - H^-)\,\delta t^-, \quad (\delta q_i^- = \delta q_i^+, \quad \delta t^- = \delta t^+) \tag{3.1}$$

Substituting the equations (1.12) into (3.1) and integrating with respect to α, we obtain the following expressions for the contours C^0 and C^1:

$$\oint_{C^0}\Big[\sum_i p_i\delta q_i - H\delta t\Big] = \oint_{C^1}\Big[\sum_i p_i\delta q_i - H\delta t\Big] \tag{3.2}$$

Thus we see that the value of the integral Poincaré—Cartan invariant is preserved when the phase coordinates of the given type undergo discontinuities.

As an example of a system with discontinuities in the phase coordinates we shall consider the instant of transition of a natural system from one region of the state space to another region with different generalized potentials. The generalized potential is given (see e.g./1/) by an expression of the form

$$V\left(q_1, \ldots, q_n, \frac{dq_1}{dt}, \ldots, \frac{dq_n}{dt}, t\right) = \sum_{i=1}^{n} A_i \frac{dq_i}{dt} + A_0$$

where A_i $(i = 0, 1, \ldots, n)$ are functions of the generalized coordinates and time t.

The generalized impulses are given by the relations

$$p_i = \frac{\partial L}{\partial (dq_i/dt)} = \frac{\partial T}{\partial (\theta q_i/dt)} - A_i \quad (i = 1, \ldots, n)$$

(T is the kinetic energy) from which it follows that the first order discontinuities in the functions A_i lead to discontinuities in the generalized impulses. The instant of instantaneous change in the generalized potential is characterized by the conditions

$$p_i^+ - p_i^- = -A_i^+ + A_i^- \quad (i = 1, \ldots, n)$$

If a function $\Pi(q_1, \ldots, q_n, t)$ exists for the differences in the right-hand sides of these equations and is such that

$$A_i^- - A_i^+ = \partial\Pi / \partial q_i \quad (i = 1, \ldots, n)$$

then all previous arguments and conclusions hold.

We find that not only the integral invariant, but also the generalised forces and hence other mechanical quantities, are invariant under the transformation of the generalized potential in question. Using the terminology employed in the field theory, we shall call the function A_0 the scalar potential and $(A_1, \ldots, A_n)'$ the vector potential. The resulting non-uniqueness of the potentials enables us to choose them in such a manner that the scalar potential vanishes. To do this, it is sufficient that the condition

$$A_0 - \partial\Pi / \partial t = 0$$

holds (when the action is impulsive, the condition can be fulfilled directly after the impulse is terminated).

The property shown above is a generalization of the property known in the field theory /3/ as the gage (or gradient) invariance of the physical quantities under the same transformation of the Lorenzian force field potential /1/.

In conclusion we note that the role of the scalar and vector potential can be played by the terms of zero form and by the set of the coefficients of linear form relating to the generalized velocities, in the expression for the Lagrangian function of the systems with non-stationary constraints.

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